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THE ONE/MANY PROBLEM IN THE FOUNDATIONS OF SET THEORY

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The question of whether the universe of set theory is a One or a Many is discussed. It is argued that although there may be a single concept of "set", the universe of set theory is not any one fixed extensional object. A streamlined form of Takeuti's Nodal Transfinite Type Theory (NTT) is presented as providing a means for making statements about the various higher-order concepts. Several new relative consistency results involving NTT are obtained. It is also shown that if NTT has a model of a certain natural kind, then $0^{\#}$ exists. A new theory called Embedding Theory (ET) is introduced as a natural strengthening of NTT. NTT is finally presented as the first theory which is about concepts as well as about sets.

INTRODUCTION

There is no fixed collection which is the class of all sets. The universe of set theory is a Many which cannot be a One. It is, however, undeniable that the single concept of "set" exists...as a sort of name with no referent. The problem I have treated is how to understand sentences in which such names appear.

It is crucial that the concept of set be treated as something quite essentially different from any set-like extensional object. For otherwise, what one thought was the universe of set theory turns out to be another set, and one's hopes of transcending set theory to contemplate the Absolute Infinite are dashed again.

Insofar as the concept of set is not represented by any fixed extensional object, it seems possible that ordinary set-theoretic statments about this and other such concepts do not have a truth value in any absolute sense. However, as Reinhardt and others have pointed out, much of our set theoretic intuition seems to arise from talking about the class of all sets.

I have endeavored to show that the question of how to talk about objects which do not strictly speaking exist is satisfactorily resolved by Takeuti's NTT.

The classical One/Many problem of metaphysics is described in §1. In §2 I relate the concept of "set" to the concept of "form of a possible thought" in order to bring out the affinity between the universe of set theory and the metaphysicians' Absolute. §3 contains some heretofore untranslated remarks by Cantor on the infinite, the Absolute Infinite, and the difficulty in discussing the latter. In §4 the exact way in which the Absolutely Infinite universe of set theory fails to be a One is brought out by means of a dialogue between a Monist and a Pluralist.

§5 is the mathematical core of this paper. Here it is shown that Takeuti's NTT is equiconsistent with a simple theory of indiscernibles; and it is shown that if NTT is slightly strengthened, then we can get $0^{\#}$ out of it. I introduce a new theory ET, which says simply that there is a non-trivial external elementary embedding of the universe into itself. ET, which is proved to be equiconsistent with NTT, is an interesting theory since, in the light of Kunen's Theorem, it sets the stage for a new set theory without the power-set axiom.

In §6 there is a discussion of the way in which NTT resolves some of the issues raised in the first four sections.

I recognize that some of my fellow mathematicians are impatient with the sort of considerations in those four sections. But it is important that set-theorists should attack the deep foundational questions --- for if all such investigation is left to mathematical amateurs, we can hope for little more than yet another discussion of what "1 + 1 = 2" really means.

I am grateful to G. Takeuti and to K. Gödel for early conversations on the topics of this paper; and to W. Reinhardt for his valuable questions about my ideas as presented in [Ru 4]. I would also like to take this opportunity to thank the faculty of the Mathematical Institute for organizing a most enjoyable conference.

§1. THE ONE/MANY PROBLEM

There are two forms of the One/Many problem: i) How many kinds of things are there? ii) How many things are there? The natural first answer is that there are many different kinds of things and many different things.

There is, however, a perennial desire to reduce the world's diverse phenomena to a single basic kind, to believe that ultimately all things are built of the same kind of stuff. Matter, sensation, thought and form have all been candidates for Urstoff. The belief that there is ultimately only one kind of thing in the world is called monism of kinds. Materialism and idealism are both monisms of kinds; the monism of kinds which asserts that everything is a set will be considered in the next section.

Instead of uniting things from the bottom up, one can work from the top down, starting with the assertion that "All is One". Monism of substance asserts that everything is a part or manifestation of a higher unity which is usually called

the Absolute.

It is, of course, obvious that the word or concept "Everything" serves to at least superficially form the world into a One. In the same way, the bare concept "set" makes a One of the universe of set-theory, but without answering the real question of whether this universe is in any sense a definite completed object. The heart of question ii) is whether or not the world is One in some organic sense, rather than in the merely syntactic and syncategorematic sense just mentioned.

Perhaps the principal reason for believing that the world is an organic One is the sort of mystical insight which Lovejoy somewhat slightingly refers to as "monistic or pantheistic pathos", ([L], p. 12). The fact that it is occasionally possible to feel an all-encompassing unity in the world is, however, not conclusive --- as it is equally possible to feel a diversity in the world which defies unification.

It is possible to argue for monism of substance in various ways. One idea is that ultimately everything in the world is related to everything else, and that the Absolute is the means or essence of this interrelatedness. Here the Absolute serves as a sort of connective tissue which fixes the individuals of the world into their perceived relational structure.

Another approach is to argue that any two things are, in a sense, the same; and that the Absolute is the one endlessly diversifying thing that exists. The essential move involved here in proving, say, that you and I are the same person is to point out that in order to express our individuality we both say the same thing: "I am I".

A problem with the more extensional monisms of substance is that if every thing is united in the Absolute, which is itself a fixed and definite thing, then the Absolute must be a component of itself --- which seems paradoxical.

This paradox is, indeed, quite real in set theory, where one is forced to sacrifice one of the three propositions: a) no set is an element of itself (genetic formation of sets); b) every mathematical object is a set (monism of kinds); or c) there is a fixed universe of all sets (monism of substance). In the usual GB or MK approach, one tries to abandon b), since there seems to be a sense in which the universe of set theory is bound together by the concept of "set". Unfortunately, it is in practice quite difficult to avoid treating the purported class of all sets like a large set. And if one's"class of all sets" is always just a large set in some better "class of all sets", then it is evident that there really is no fixed and completed universe of set theory. So unless one is very careful to avoid a assuming some form of b), then one always ends by dropping c). (On p. 9 of [R1] there is an example of a particular theory about the class of all sets being discussed first in terms of dropping b), and then in terms of dropping c).)

The option of dropping a) is not really viable for the set-theorist; not only because a) is essential to the classical conception of set, but also because

paradox can be avoided by dropping a) only if certain restrictions on the comprehension principle or on the underlying logic are adopted as well.

The philosopher Josiah Royce does take this approach in his essay, "The One, the Many and the Infinite", ([Ro], pp. 473-588), by asserting that the Absolute is what he terms a self-representative system. He uses the interesting analogy of a perfect map of England which is to be drawn on a field in England. Being perfect, this map includes an exact replica of itself, which includes an exact replica of itself, ad inf. Ideally my consciousness also constitutes a self-representative system --- assuming, that is, that one of the things in my consciousness is an exact image of my consciousness. Whether or not this is the case is, of course, debatable.

The difficulty with self-representative systems is that they cannot be directly described or built up by any step-by-step process. They must be grasped all at once, or not at all. It is at least questionable whether such an Absolute actually exists, and pluralism of substance remains a reasonable position.

This position is forcefully presented by William James in A Pluralistic Universe: "...the pluralistic view which I prefer to adopt is willing to believe that there may ultimately never be an all-form at all, that the substance of reality may never get totally collected, that some of it may remain outside of the largest combination of it ever made, and that a distributive form of reality, the each-form, is logically as acceptable and empirically as probable as the all-form commonly acquiesced in as so obviously the self-evident thing." ([J], p. 34).

In terms of set theory this is the view that there really is no class of all sets. There are only variously large sets --- any one of which may be falsely assumed to be the whole universe. When people speak of the class of all sets they can never be correctly referring to any specific collection; instead they are referring to the syntactic concept of set. My sympathies lie with this pluralistic position, and I will discuss in §6 how the theory NTT can account for the fact that although there really is no fixed extensional class of all sets, it is possible for statements about the class of all sets (such as "Ω is a measurable cardinal") to be meaningful for us.

§2. WHAT IS A SET?

Most simply, a set is a "Many which allows itself to be thought of as a One," ([C], p. 204). A set in this classical sense is, as Cantor remarks (ibid), a sort of little Absolute, a partial world of individuals which are combined into a One by a creative act of thought.

It may seem strange that the definition of set has a psychological component, but the concept of "set" has been related to the concept of "possible thought" from the very beginning.

For instance, in his 1887 essay, "Was sind und was sollen die Zahlen?", Dedekind proved the existence of infinite "systems" by pointing out that his thought-world (Gedankenwelt) was infinite since there is a map which injects the Gedankenwelt into a proper part of itself. This is the map which takes a thought s into the thought "s is a possible thought", (see [D], p. 64).

This proof is not conclusive since, as Cantor mentions in passing in an 1899 letter to Dedekind, the "totality of everything thinkable" is an absolutely infinite or inconsistent multiplicity such that "the assumption that all its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'." (see [C], p. 443, or the translation [CD], p. 114).

But why is the Gedankenwelt an inconsistent multiplicity? The essential assumption is that our thoughts are not self-representative systems; that is, that a thought does not include itself as an object. Therefore when I try to form my Gedankenwelt into a One, I am producing a new thought which has not been accounted for. Including this new thought produces a synthesis which constitutes a further new thought, and so on. (See §1 of the paper abstracted in [Ru3] for a more detailed discussion of this process.)

Incidentally, Dedekind's proof was derived from Bolzano's proof that there are infinite sets: "The set of all true propositions is easily seen to be infinite. For if we fix our attention upon any truth taken at random..., and label it A, we find that the proposition conveyed by the words 'A is true' is distinct from the proposition A itself..." ([B], pp. 84-85). This proof is flawed in the same way that Dedekind's is, since the totality of all true propositions is in fact an inconsistent multiplicity. For if the notion of truth were definite, graspable and named by the word 'truth', then the sentence "This sentence is not true" would be meaningful --- which is manifestly impossible.

So we see that when Cantor speaks of a set as a Many which allows itself to be thought of as a One, he is using "thought" in the sense of "non-self-representative, rationally communicable thought". In other words, a set is to be a (rationally) conceivable form.

In view of the fact that, by the Completeness Theorem, any conceivable structure can be modelled set-theoretically; it seems permissible to assert, in the spirit of §2.1 of [W], that any fact, thought, or object has a "picture" which is a set. Once we have gone this far it is tempting to jump to a full monism of kinds, saying that every conceivable form is a set.

This jump is to be accomplished by identifying any given form with one if its set models. This process is not so arbitrary as it initially appears, for once we have modelled some system by assigning sets as labels which stand for the various individuals and relations involved, and by setting up the appropriate artificial membership relation...once we have such a set model it can be transformed by the

Mostowski collapsing technique into a set model with the standard membership relation. The only element of arbitrariness in this final model is the choice of labels for the atoms and non-well-founded portions of the original system. (The approach just outlined could possibly be used to refute the claim by Benacerraf and others that there is no distinguished set-theoretic model of the natural numbers.)

Supposing that one accepts the identity between the concepts "set" and "conceivable form", can one then say that every thing is a set?

The primary objection to this is that there seem to be things which are not conceivable forms, e.g., the class of all sets or the totality of all conceivable forms. One might simply insist that these inconsistent multiplicities do not exist in any sense --- and it is true that they do not exist as completed extensional objects --- but it still seems that they do have some sort of existence, since it is possible to talk about them.

Is it at least legitimate to regard the various individual objects in the world as sets? Certain monists of substance would say not, arguing that in order to fully express all the aspects of any given thing it is necessary to bring in everything, so that no individual thing would embody a conceivable form after all.

There are two other weak points in the view that everything is a set. First of all, the experienced fact that things are one way and not another, that I am myself and the world is this world...this sort of particularity does not seem to be provided for by saying that I am a certain point in a certain complex relational system. That is, there does not seem to be any way to represent set-theoretically the fact that it is this world which really exists. This objection could be countered with the claim that every possible world really exists.

I will insert here a brief digression on physical monism of substance. As Gödel pointed out in [G1], the spacetime viewpoint of relativity theory seems to provide an excellent vindication of the idealistic view that the passage of time is illusory, that the universe does not fall into many distinct "nows", and that spacetime is one in substance. This physical monism of substance has been shaken by recent interpretations of quantum mechanics which, in order to account for indeterminism, contemplate the existence of every possible physical universe, (see [DG] and Chapter 43 of [MTW]). In these models the unified aether-like spacetime of relativity is split into many distinct spacetimes. Whether one can move through such a viewpoint to a higher monism of substance which asserts that all the possible universes are points in a static "superspace" is problematic because a) the existence of such a superspace as a One is exactly as debatable as the existence of the collection of all possible thoughts, and b) it seems possible that there could be various sorts of superspaces existing as points in a supersuperspace, etc.

To return to the main line of argument, a second, related, objection to the view that everything is a set is that the set-theoretic model does not seem to account for the fact that the world is going on. John Wheeler speaks of this

difficulty as it relates to an imagined room full of equations intended to represent the physics of the universe, "Stand up, look back on all those equations, some perhaps more hopeful than others, raise one's finger commandingly, and give the order 'Fly!' Not one of those equations will put on wings, take off, or fly. Yet the universe 'flies'." ([MTW], p. 1208). This objection could perhaps be met by the assertion that there is nothing more to the "life" of the world than the various forms and formations which occur.

But the question of whether or not the objects of the everyday world are in some sense really sets is not central to this paper. I have raised it here only in the hope of stimulating further discussion; and to remind the reader of the close similarities between the universe of set theory, the universe of thought, and the universe of physics.

§3. THE ABSOLUTE INFINITE

The viewpoint that every conceivable form is a set is a modernization of the Pythagorean doctrine that "all things are numbers," (see [H], p. 67). The Pythagorean doctrine in its original form was invalidated by the discovery of the existence of continuous magnitudes whose ratios were not equal to that of any two natural numbers. A pluralism of kinds as regarding mathematical entities seemed unavoidable. Indeed, so far were such medieval mathematicians as Vieta from the present mathematicians' set-theoretic monism of kinds, that they viewed it as impossible to add magnitudes which represented lengths to magnitudes which represented areas.

Dedekind and Cantor showed that irrational magnitudes could be precisely represented by infinite sets based on natural numbers. As far as Cantor was concerned there was no question that the only way to mathematize the continuous was by means of infinite sets. "One can without qualification say that the transfinite numbers stand or fall with the finite irrationals; their inmost essence is the same, for these are definitely laid out instances or modifications of the actual infinite." ([C], pp. 395-396.)*

There are of course those who would deny that irrationals exist as actual infinities; those who would rather suffer an avoidable pluralism of kinds and view irrational numbers as being best represented by idealized computing devices which

*I have provided the translations from the German which appear in this section. I recently learned that W. Reinhardt has been independently preparing a paper which includes his own translations and comments upon some of the same passages from Cantor as those which appear here.

produce, given any number n, the $n^{\underline{th}}$ place of some decimal expansion. There is a spirited passage in "Uber die verschiedenen Standpunkte in bezug auf das aktuelle Unendliche" where Cantor attacks such "Horror Infiniti" as "...a form of myopia which destroys the possibility of seeing the Actual-Infinite, even though it in its highest form has created and sustains us, and in its secondary transfinite forms occurs all around us and even inhabits our minds." ([C], p. 374).

Going out from a neo-Pythagorean belief that everything must be mathematically representable by some static form, Cantor actually takes the existence of continuity as evidence for the existence of acutal or completed infinities. But what is the "highest form" of the infinite which he alludes to?

He distinguishes three kinds of actual infinities, "The actual infinite arises in three contexts: first when it is realized in the most complete form, in a fully independent other-worldly being, in Deo, where I call it the Absolute Infinite or simply Absolute; second when it occurs in the contingent, created world; third when then mind grasps it in abstracto as a mathematical magnitude, number, or order type. I wish to make a sharp contrast between the Absolute and what I call the Transfinite, that is, the actual infinities of the last two sorts, which are clearly limited, subject to further increase, and thus related to the finite." ([C], p. 378).

Later, Cantor amplifies, "The Transfinite with its richness of forms and formations points necessarily to an Absolute, to that 'true infinitude' whose magnitude admits of no increase or decrease, and which is therefore to be quantitatively viewed as an absolute maximum. This Absolute is in certain ways beyond the grasp pf the human mind, and does not admit of a purely mathematical representation..." ([C], p. 405)

This last sentence seems to echo what a Pythagorean might have said about the incommensurable. Is there any hope of carrying out a mathematical treatment of the Absolute, or must such investigations remain, as Cantor suggested, in the province of speculative theology? Perhaps not --- recall Russell's response to §7 of the Tractatus, "...after all, Mr. Wittgenstein manages to say a good deal about what cannot be said..." ([W], p. xxi).

It seems clear that, just as mathematics had to move beyond number theory in order to treat the incommensurable, mathematics would have to move past set theory in order to treat the Absolute. The analogy is, however, slightly misleading. The Pythagoreans could not really question the existence of incommensurable magnitudes. But it is perhaps possible for a set-theorist to assert that there is really no Absolute universe of set theory. I agree with this up to a point --- but it does seem to be undeniable that i) we do have a definite concept of set, and that ii) it would be nice to have a framework in which such concepts could be treated like real objects.

§4. A DIALOGUE ON SETS AND CONCEPTS

The class of all sets, C, is a perfect example of Cantor's Absolute Infinite, as it is clearly unvermehrbar (not subject to augmentation). Moreover, if every conceivable form is, or codes up, a set (as was suggested in §2), then C is very close indeed to the Absolute of the metaphysicians.

The question I have been leading up to is this: Is C a One or a Many? I will approach this question and some of its ramifications by way of a dialogue between a Monist and a Pluralist.

P: For all the familiar reasons it is evident that C is not a set. Therefore C is not a Many which allows itself to be thought of as a One. So the universe of set theory is, and must remain, a Many.

M: But you just referred to the universe of set theory, did you not? Is it not evident that the concept of "set" serves to bind all the sets into a single vast entity which we call C?

P: When you speak of the concept of "set" you are in fact unconsciously harboring a model of this concept. Your model, being a thinkable form, is a set. Any individual's concept of the class of all sets is in fact just a set. There are only the many illusory universes and no correct one.

M: I was not referring to some limited individual's concept of "set". I wish rather to direct your attention to the absolute and independently existing concept of "set" which lies beyond our imperfect realizations thereof. Surely you can decide for any given entity whether or not it is a set?

P: I can also decide for any given entity if it is an entity or not. But this says nothing! Who is it that gives me entities? A set is not always presented like "a carefully folded ham-sandwich". A set is essentially the inner form of a mathematical structure --- and it has happened to me more than once that I could not immediately discern the form which some given mathematical description embodied. Someone who has never heard of ultrafilters cannot pretend to know, in any real sense of the word, whether or not the concept of "set" includes them.

M: You are again basing your argument on individual inadequacies. The concept of "set" is objective and external to us. The fact that sets are perceived through the mind does not mean that they depend on minds for their existence. In the same way, objects exist without hands to feel them, sights exist without eyes to see them, and all the possible thoughts exist independently of any thinkers.

P: How can you say that a thought can exist independently of any thinker?

M: Any thought which an individual perceives can, in principle, be perceived by someone else. And, as Einstein puts it, "We are accustomed to regard as real those sense perceptions which are common to different individuals, and which

therefore are, in a measure, impersonal." ([E], p. 2). Given that the same thought can be perceived by different people at different places and times, it is a natural simplification to assume that thoughts have an objective existence external to us. The universe of thoughts is a One, and all the sets --- which are simply the forms of possible thoughts --- constitute a One as well.

P: I can agree that the individual sets exist objectively and independently of us. But I still maintain that there is nothing to simultaneously grasp them all and pull them together into a One. You insist that the concept of "set" does this, but I still argue that there are so many different kinds of sets that there is, in fact, no such graspable concept.

M: I never said that you could grasp the class of all sets. I only say that the bare notion of set as something built up from the empty set by iterated applications of the operation 'set of' is perfectly clear. Since we have the name C under which all the sets fall, it seems undeniable that the class of all sets is a One.

P: Suppose I grant you this point for the sake of further argument. We must agree, however, that your concepts are essentially syntactic rather than semantic objects. "C" is a name, a defining property, not some completed extensional object. For if C were a static collection, then it would be essentially a set... which is impossible.

M: That seems fair. But I don't like having two kinds of mathematical objects: sets and concepts. Perhaps we should view sets as special sorts of concepts.

P: You may, but I prefer not to. Once you start lumping sets and concepts together it is easy to start thinking of concepts as fixed extensions, which they are not.

M: Nevertheless, I think that the $\in$-predicate can be meaningfully extended to the concepts.

P: This worries me. There are many inconsistent multiplicities of sets which can exist as concepts. For instance, the class of all ordinals and the class of all models of ZF seem to be represented by the concepts "ordinal" and "set model of ZF". Now why not view $\wp C$ as existing as the concept, "concept which applies only to sets". And then why not continue to get a new enlarged set-theoretic universe on top of what we thought was the unvermehrbar universe of set theory?

M: You yourself said that the concepts are not fixed extensional objects, not sets. A concept is not a collection, but rather a name for a sort of contemplated activity. We can have all of these objects above the universe of set theory, but they are not sets.

P: And what comes after all the concepts?

M: The concept of "concept". This self-representative system does not lead to a Russell paradox because it is not assumed that for any two concepts A and B,

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the question "Does concept A fall under concept B?" has an answer; any more than does the question "Is triangularity virtuous?". Gödel supplies a discussion of this move in [G2], pp. 228-229.

P: Nevertheless, I think that your belief in the reality of a concept of all concepts is misplaced. For one thing, you have not given any explanation of what is meant by the concept of "concept"...perhaps because any such explanation could be automatically transcended? Primarily it seems to me that to assert that there is a concept of all concepts is false in the same way that it is false to assert that there is a nameable function which transforms any string of symbols into the set, if any, named by the string in question. If a concept is a syntactic object, then you cannot expect there to be a concept of all concepts, for this would be a syntactic object which encompasses all syntax. Just as Russell's paradox shows that there is no set of all sets, Richard's paradox indicates that there is no concept of all concepts.

M: That's a good argument if you believe that concepts are basically just names. It interesting to note that even if every concept were nameable, you could never really know this...since your notion of the naming process will always be inadequate.

P: There seems to be a little fuzziness in the statement that every concept is nameable. For, the way I see it, we start with a lot of names with set parameters, then call them concepts and treat them like objects, then figure out an $\in$-theory for these objects, and finally check that the theory is such that each of the names behaves as an object having the property specified by its name. How do we decide what the $\in$-theory is? And why should the process I just described come out right?

M: First, in response to your second question, I'm not sure that the process you describe should work out at all. Even if every concept is nameable, it will not necessarily be by one of the names in the language you start out with. Indeed, relative to any of the nameable languages which you use, there always will be unnameable concepts. Second, in response to your question of how one discovers the $\in$-theory of the concepts, I would say that this is to be found by a process of thinking about the nature of the concepts in question.

P: I can see that such an approach might enable you to determine the truth value of certain very simple statements about concepts. But on what basis are you going to decide if $\mathcal{P}C$ is well-ordered? Your approach of "thinking about the concepts" is just an incomplete proof-theoretic procedure.

M: There is no guarantee that my thought processes would not, in the ideal limit, generate a complete theory. A person's thoughts are not an r.e. set, but rather what Myhill calls in [M] a "prospective character". I would say that thinking about the concepts leads towards a complete theory, and that this theory determines the true set theory.

P: Why would the theory of concepts have anything to do with the theory of sets?

M: The reflection principle ties the two together. If I discover that C has some property, then I know that many R_α have the same property. If I learn that $R_{\Omega+\Omega}$ behaves a certain way, then I know that many $R_{\alpha+\alpha}$ behave the same way... Ω being the ordinal of C. It is the behaviour of the higher-order concepts which produces, via the reflection principle, the harmonies which obtain within C. I might remark here that it may be that true axioms of infinity are just those insights into set theory which are gained by meditation upon the nature of concepts. If this is the case, and if it is the case that in the ideal limit one finds a complete theory of concepts, then Gödel's 1946 conjectures are true. (See [G3] and [Ru2]).

P: Let's pause here. We both agree that C is not a fixed extensional object, but is rather a concept...although we aren't too sure what a concept is. But where we differ is that you seem to believe that there are certain considerations which will make it evident what the $\in$-theory of the well-founded concepts over the sets is.

M: And it is this ultimate $\in$-theory which regulates the theory of sets. The individual sets can be viewed as emanating from, or arising in imitation of, the realm of Absolutely Infinite concepts.

P: I continue to doubt that there is any definite $\in$-theory on the concepts. I think that when you act as if there is, you are simply falling back into the lazy habit of treating concepts as fixed extensional objects. Rather than make this mistake, I would prefer to believe that concepts are purely syntactic entities, names with no referent. Any $\in$-theory which we place on them arises only by way of some convention which we adopt. I would like to direct your attention to a pluralistic theory which determines the behaviour of the essentially fictional concepts from the behaviour of sets, rather than vice-versa. This theory, called Nodal Transfinite Type Theory, or NTT, was invented about ten years ago by Gaisi Takeuti.

§5. NTT

Most of the highly technical section 5 is omitted here.

In [Da] and [T], NTT is formulated over the language TT of "transfinite type theory". TT differs from the language of ZF in that it has infinitely many different types of variables. In general, if $A(v_0,v_1)$ is a TT-formula, then there are, in TT, variables x^A of type A which are intended to range over $R_{(\mu\alpha)[A(\Omega,\alpha)]}$.

It is simpler to formulate NTT over the language TN of what we might call "trans-set name theory". TN is the smallest language extending the language of ZF and having the property that if $A(v_0,v_1)$ is a TN formula, then there is an individual constant @A in TN. Let me remark here that "language of ZF" is used in the broad sense under which all the usual defined symbols are allowed.

$$\leftrightarrow \lambda_0 \in \{\theta \in j'\lambda_0 : (\forall x \in R_\theta)[\theta \in (j'f)'x]\}$$
$$\leftrightarrow \{\theta \in \lambda_0 : (\forall x \in R_\theta)[\theta \in f'x]\} \in \mathfrak{N}.$$

C) is proved as follows. Let ψ be a ZF-formula, and let $x \subseteq R_{\lambda_0}$.

$$\psi[x,\lambda_0,\ldots,\lambda_n] \leftrightarrow \lambda_0 \in \{\alpha \in j'\lambda_0 : \psi[j'x \cap R_\alpha, \alpha, \lambda_1, \ldots \lambda_n]\}$$
$$\leftrightarrow \lambda_0 \in j'\{\alpha \in \lambda_0 : \psi[x \cap R_\alpha, \alpha, \lambda_0, \ldots, \lambda_{n-1}]\}$$
$$\leftrightarrow \{\alpha \in \lambda_0 : \psi[x \cap R_\alpha, \alpha, \lambda_0, \ldots, \lambda_{n-1}]\} \in \mathfrak{N}. \blacktriangle$$

What sort of large cardinal consistency results of the form Con(ZF + $(\exists \alpha)\psi[\alpha]$) can we extract from Con($\overline{\text{NTT}}$') ?

In [AHKZ] a family of combinatorial large cardinal properties called "flipping properties" is introduced. In [Z] the particular flipping property of being a "λ-piecewise strongly compact cardinal" is defined. I will not give the rather involved definition here.

The existence of these particular large cardinals is an internal property of the universe, but Zwicker shows that their existence also follows from the type of external properties of the universe considered by NTT and ET. In particular, he says that for a model M of ZF with $\kappa, \lambda \in M$, $j \in {}^{M}M$ is a κ-λ-outside embedding if $M \simeq j''M \prec M$, if κ is the first ordinal moved by j, and if $j'\kappa > \lambda$; and he shows that if there is a κ-λ-outside embedding, then κ is λ-piecewise strongly compact. From these considerations we obtain a proposition.

PROPOSITION 10 (ZF) If there is a transitive model of $\overline{\text{NTT}}$', then there is a transitive model of ZF + $(\exists \kappa)(\forall \lambda)[\kappa$ is λ-piecewise strongly compact].

Pf: The proof is much the same as that of Theorem 9.i. If we first cut U down to $\cup\{R_{\lambda_n}(U) : n \in \omega\}$ there, and then continue as usual, the λ_n's will be cofinal in T. By extending the map $j'\lambda_n = \lambda_{n+k}$ to all of T, for various k, we can get all the external embeddings we need. $\blacktriangle$

§6. THE MEANING OF NTT

In the first four sections of this paper I developed the point of view that i) The class of all sets C is a concept which can and should not be viewed as a definite collection; ii) It is necessary for the advance of mathematics to find some way to interpret sentences of the form $\psi[C]$ whose quantifiers range over some concept, or non-extensional entity, which lies beyond C; and iii) It may be reasonable to assume that every concept has a name which it can be identified with.

Now, the reflection principle $\psi[C] \rightarrow \mathfrak{N}(\{\alpha \in \Omega : \psi[R_\alpha]\})$ expresses the feeling that the behaviour of the concepts produces the observed harmonies in the universe of sets. One of the ideas underlying NTT is to reverse this implication and also say that $\mathfrak{N}(\{\alpha \in \Omega : \psi[R_\alpha]\}) \rightarrow \psi[C]$.

The attempt to pass from knowledge of the truth values of all the $\psi[R_\alpha]$'s to the truth value of $\psi[C]$ is somewhat analagous to the way in which one attempts to pass from knowledge of all the partial sums of an infinite series to the sum of the whole series. According to which convention one adopts, the Grandi series 1-1+1-1+ 1-1+... can add up to 0, 1/2, 1, or something else. The fact that this series has a sum only by convention is overlooked sometimes in order to formulate such puzzles as the Thompson Lamp.

One is tempted to say that, insofar as there is no fixed extensional entity which is the class of all sets, the way in which one passes from knowledge of the α's for which $\psi[R_\alpha]$ holds to a decision on the truth value of $\psi[C]$ is also purely conventional. NTT explicitly introduces one such a convention in the form of the primitive concept of nodality.

To take the analogy,between talking about concepts and finding the sum of an infinite series,a bit further, note that for a convergent series there is only one summation convention which seems acceptable. It is worth recalling, however, that there have been other conventions for determining the sum of even these infinite series --- I am thinking here of the Zenonian convention that any infinite series of positive terms has infinite sum.

In the same way, NTT may very well embody the most acceptable convention for interpreting talk about the class of all sets, but this does not mean that other conventions are impossible.

For instance, there are some set-theorists who always work in the theory MK. They are always interested in natural models of the form $R_{\theta+1}$, for θ a strong inaccessible. Is it not fair to say that for these set-theorists, the class $\{\theta+1: \theta$ is strongly inaccessible$\}$ is "nodal", that is, comprised of "typical" ordinals? And can one not then go on to say that for these set-theorists, $C \models$ "There is a greatest ordinal" ? You may object that there can be no greatest ordinal, but I submit that for someone who always thinks in terms of MK, there is a greatest ordinal, i.e. "On".

The axioms in group B) of NTT can be viewed as formalizing the intent that our truth convention should be so as to make statements about C behave like statements which are actually about some specific model of ZF. B2) forces statements about C (i.e. $R_{@0}$) to obey modus ponens. B3) allows us to pass from "For every set x, $\psi[x,C]$ holds" to "$(\forall x \in C)[\psi[x,C]]$ holds". If we are working with NTT', then it is B3) which ensures that C is a model of ZF.

Suppose that we adopt the view that every concept has a name. Of course, no single theory which I can actually refer to will name all of the concepts --- for to refer to a theory is to provide a name for it, and this name can be used to lump together all the concepts named by the theory, yielding a concept not named by the theory. I feel that a theory is only able to say things about the concepts it can name. Looking at Theorem 5.ii with this in mind, one can see that $\overline{\text{NTT}}$' is really just as good a theory as $\overline{\text{NTT}}$.

The λ_n constants are used in the language of $\overline{\text{NTT}}$ and $\overline{\text{NTT}}'$ in order to name some concepts which are not named by the language of ZF. For instance, λ_1 names a concept which lies beyond all concepts of the form $\tau[u,\Omega]$ for τ a ZF-term and u a set. To accept the name or concept "λ_1" requires a leap of faith not much greater than that required to accept the concept variously referred to as "On", "@0", "λ_0", and "Ω".

The only author I know of who has yet had occasion to discuss the concept λ_1 is W. Reinhardt, who refers to it as "λ" on p. 200 of [R2]. Reinhardt here describes how the inability to see that there are concepts beyond λ_1 can lead one to believe that there is an internal elementary embedding from $R\lambda_1$ into $R\lambda_1$.

I might mention now, that once we are familiar with NTT, we are in a position to move beyond it to include yet more named concepts in our domain of discourse --- for instance we can try to get a theory which includes $\{x: \mathfrak{N}(x)\}$ and $\{\lambda_n: n\in\omega\}$ in its world. It is interesting to note that by Corollary 7.ii, conceiving of the latter concept enables one to conceive of a non-constructible set of integers. In my opinion this is the best justification for $V\neq L$.

To get back to NTT, viewed as an actual theory about all the sets and some of the concepts, why exactly should Schema C) hold?

What does it mean to say $\psi[\Omega] \leftrightarrow \mathfrak{N}(\{\alpha \in \Omega: \psi[\alpha]\})$? This is a statement about the concepts $\psi(\ )$, Ω, and $\mathfrak{N}$. On the left-hand side we have a statement that the concept Ω falls under the concept ψ, and on the right-hand side we have the statement that the concept $\{\alpha \in \Omega: \psi[\alpha]\}$ is nodal. Since ψ is not relativized on the right, and since concepts occur on both sides of the biconditional,it would be misleading to say that Schema C) reduces talk about concepts to talk about sets and the single concept of nodality. Indeed, Schema C) accomplishes no reduction at all when applied to a ZF-sentence σ (for $\mathfrak{N}(\{\alpha\in\Omega: \sigma\})$ is already equivalent to σ without any use of Schema C)) --- this is not surprising since it is only in axiom group A) that NTT makes any statements about all the concepts at once. (Note, in particular, that the schema $\sigma \leftrightarrow R_\Omega \models \sigma$ is nowhere assumed.)

Schema C) does not really provide an interpretation for statements about all the concepts at once, but it does provide an interpretation for formulae whose quantifiers are bound to some specific concept. A typical kind of formula to which Schema C) is usefully applied has the form $R_{\tau[\Omega]} \models \psi[\Omega]$, where τ is a simple functional term (i.e. $\tau[\Omega]$ is $\Omega+1$, $\Omega+\Omega$, Ω^+, $(\mu\theta)[R_\Omega \prec R_\theta]$, or something like that). Then we have $R_{\tau[\Omega]} \models \psi[\Omega]$ iff $\mathfrak{N}(\{\alpha\in\Omega: R_{\tau[\alpha]} \models \psi[\alpha]\})$. This is definitely a reduction of a sentence about concepts of rank greater than Ω to a sentence about $\mathfrak{N}$ and a concept of rank Ω. Insofar as we feel that our idea of the behaviour of Ω is arrived at by abstracting from the behaviour of all the α's, the form of this reduction certainly seems reasonable.

The way in which Schema C) applies to formulae involving the λ_n's is particularly attractive. The plausibility of $\psi[\Omega,\lambda_1,\ldots,\lambda_n] \leftrightarrow \mathfrak{N}(\{\alpha_0: \mathfrak{N}(\{\alpha_1: \ldots \mathfrak{N}(\{\alpha_n:$

$\psi[\alpha_0,\alpha_1,\ldots,\alpha_n]\}) \ldots \})\})$ comes from the feeling that the behaviour of, say, Ω relative to λ_1 is abstracted from the behaviour of a "typical" ordinal α_0 relative to a larger "typical" ordinal α_1.

To sum up, the situation leading to NTT seems to be as follows: 1) Each set exists as a definite extensional object. Beyond the sets there are concepts such as C (the concept of "set"), Ω (the concept of ordinal), $\mathcal{P}C$ (the concept of concept applying only to sets), and so on. 2) We think of each concept as eventually having a name. To make our naming ability somewhat richer than that of the language of ZF, we add for each n a name λ_n, with the understanding that each λ_n represents a new degree of unnameability. 3) A partial $\in$-theory on the concepts can be derived from syntactic considerations. E.g., it is evident that if $y \in \mathcal{P}C$ and $x \in y$, then $x \in C$. However, since the concepts are not fixed extensional objects, they do not seem, a priori, to possess a complete $\in$-theory as the sets do. 4) Given a statement about concepts, like $\psi[\Omega,\lambda_1]$, we test its truth by looking at all sentences of the form $\psi[\alpha,\beta]$. If there is an "overwhelmingly dense" or nodal class of α's, for each of which the class of β's such that $\psi[\alpha,\beta]$ is nodal, then we say that $\psi[\Omega,\lambda_1]$ holds. 5) The assumptions in axiom group B) formalize certain reasonable intuitions about the concept of nodality just used. Nodality is a primitive concept arising from the mind's ability to look at the given universe of sets and concepts from the outside.

There is one significant objection to this procedure, and this is that it does not seem to be automatically true that the $\in$-theory on the concepts derived from Schema C) will mesh with the partial $\in$-theory on the concepts which can be derived from proof-theoretic considerations of the names involved. In particular, we can prove that $\mathcal{N}(\{\theta: \theta$ is strongly inaccessible$\})$. Since AC is assumed to hold in C, this implies that $\mathcal{N}(\{\theta: (\exists w \in R_{\theta+1})[w$ well-orders R_θ with length $\theta]\})$. So by Schema C), $(\exists w \in R_{\Omega+1})[w$ well-orders C with length $\Omega]$. But can we exhibit a name of any such well-ordering of C?

A solution to this difficulty may be to use the theory NTT + V=OD which is obtained by replacing axiom group A) by group A)*: The axioms of ZF plus the axiom V=OD, (where V=OD is an abbreviation for $(\forall x)(\exists \alpha)[x$ is definable in $R_\alpha]$). This would not lead to $C \models$ V=OD; for the "ordinals" from which the members of C are defined might even lie beyond all the λ_n's. (See Illustration XI of [Ru 1] for an example of a V≠OD universe which has a well-ordering nameable from higher-order concepts.) The assumption that every set or concept can eventually be defined from some ordinal, where the ordinals are allowed to go past Ω, would seem to be in the spirit of Gödel's 1946 conjecture that every ZF-sentence is decidable from some true axiom of infinity (see [G3] and [Ru 2]). In NTT + V=OD the question at the end of the last paragraph can be answered by exhibiting the name "$(\mu^{\leq OD}w)[w$ well-orders R_Ω with length $\Omega]$".

In conclusion, NTT seems to be a reasonable theory about sets and concepts. The

great virtue of NTT is that it treats concepts such as C as different from sets --- that is, NTT does not make the mistake of treating C like some large but definite collection. So long as one makes this mistake one is still doing set theory, as opposed to concept theory. The importance of seeing beyond set theory has been stressed, in a somewhat different context, by Hájek and Vopěnka: "...the authors hope to make some contribution to the task of breaking through the bars of the prison in which mathematicians find themselves. This prison is set theory and the authors believe that mathematicians will escape from it just as they escaped from the prison of three-dimensional space." ([HV], p. 12).

It remains to draw a metaphysical moral from the preceding mathematical discussion.

On the one hand the Absolute is not merely a Many, a random hodge-podge, a relentless accumulation of particulars --- for in the very act of saying "the Absolute", we acknowledge our conviciton that the world is One. On the other hand, the Absolute is not a fixed and immutable One, for to conceive of any (non-self-representative) One is to see how to transcend it.

This dilemma can perhaps be resolved by viewing the Oneness of the Absolute qua subjective concept, and the Manyness of the Absolute qua objective non-set, as "complementary but exclusive features of the description"; and to say that, "In fact, here again we are not dealing with contradictory but with complementary pictures of the phenomena, which only together offer a natural generalization of the classical mode of description." ([Br], pp. 54-56). The analogy to quantum mechanics is not totally unnatural if we agree with Bohr that the complementarity of causal laws and space-time descriptions "bears a deep-going analogy to the general difficulty in the formation of human ideas, inherent in the distinction between subject and object." ([Br], p. 91. Cf. also the last section of [Be]).

The significance of NTT for the metaphysical One/Many problem is that it shows how it is possible to attain a rigorous logical convention connecting the two complementary views of the Absolute.

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